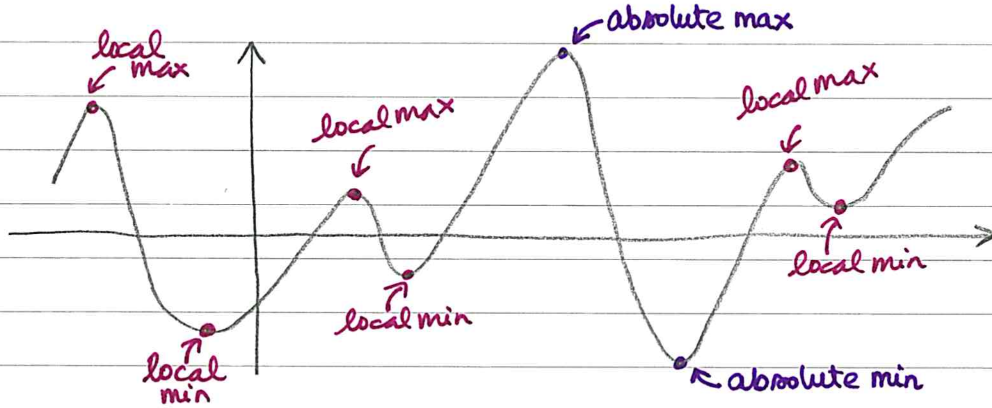
3.1. Extreme Values

Let c be in the domain D of a function f.

$$f(c) = \begin{cases} \text{absolute maximum of } f \text{ on } D \text{ if } f(c) \geq f(x) \text{ for } \underline{\text{all}}\ x \in D \\ \text{absolute minimum of } f \text{ on } D \text{ if } f(c) \leq f(x) \text{ for } \underline{\text{all}}\ x \in D \end{cases}$$

$$f(c) = \begin{cases} \text{local maximum of } f \text{ if } f(c) \geq f(x) \text{ when } x \text{ is } \underline{\text{near}}\ c \\ \text{local minimum of } f \text{ if } f(c) \leq f(x) \text{ when } x \text{ is } \underline{\text{near}}\ c \end{cases}$$



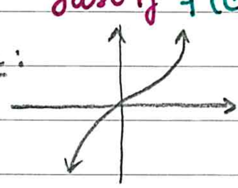
Extreme Value Theorem: If f is continuous on a closed interval $[a,b]$, then f attains an <u>absolute maximum</u> value $f(c)$ and an <u>absolute minimum</u> value $f(d)$ at some numbers c & d in $[a,b]$.

Fermat's Theorem: If f has a local min or max at c, and if $f'(c)$ <u>exists</u>, then $f'(c)=0$.

<u>Caution #1</u>: The converse is not always true, i.e.

Just if $f'(c)=0$ does not mean f necessarily has a min/max there!

<u>Example</u>:

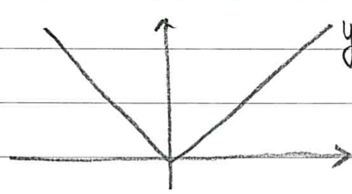


$f(x) = x^3$

$f'(x) = 3x^2 \leftarrow f'(0)=0$ but f has no min/max at 0!

<u>Caution #2</u>: There could be an extreme value where $f'(c)$ does not exist!

<u>Example</u>:



$y=|x|$ has an absolute <u>min</u> at $x=0$, but is not differentiable at $x=0$!

$\Rightarrow$ These lead to the notion of <u>critical number</u>.

Def.: A critical number of a function f is a number c in the domain of f such that either $\boxed{f'(c)=0}$ or $\boxed{f'(c) \text{ DNE}}$.

⇒ Fermat's Theorem Reworded:

If f has a local max or min at c, then c is a critical number of f.

Examples: Find the critical numbers of the functions below:

① $f(x) = \dfrac{3x}{3x^2+10}$ denom. never 0 ⇒ $f'(x)$ exists everywhere

$$f'(x) = \frac{3(3x^2+10) - 3x(6x)}{(3x^2+10)^2}$$ ← numerator has to be 0

$$f'(x)=0 \iff 9x^2+30-18x^2=0$$
$$30-9x^2=0$$
$$30=9x^2 \qquad x^2=\frac{10}{3} \Rightarrow x = \boxed{\pm\sqrt{\frac{10}{3}}}$$ c.pts.

② $f(x) = (x-1)(x-5)^3+11$

$$f'(x) = (x-5)^3 + (x-1)\cdot 3(x-5)^2$$
$$= (x-5)^2\left[(x-5)+(x-1)\cdot 3\right]$$
$$= (x-5)^2(x-5+3x-3)$$
$$= (x-5)^2(4x-8) \qquad \text{set } f'(x)=0 \Rightarrow x=\boxed{2,5}$$ c.pts.

③ $f(x) = 7x^{2/3} + 3x^{5/3}$

$$f'(x) = 7\cdot\frac{2}{3}x^{-1/3} + 3\cdot\frac{5}{3}x^{2/3} = \frac{14}{3}\frac{1}{\sqrt[3]{x}} + \frac{15}{3}(\sqrt[3]{x})^2 = \frac{14+15(\sqrt[3]{x})^3}{3\sqrt[3]{x}}$$

$$f'(x)=0 \iff 14+15x=0 \iff x=-\frac{14}{15}$$

⇒ Critical numbers: $\boxed{0, -14/15}$ (b/c $f'(x)$ DNE at $x=0$!)

④ $f(x) = x^{1/7} - x^{-6/7}$

$$f'(x) = \frac{1}{7}x^{-6/7} + \frac{6}{7}x^{-13/7} = \frac{1}{7(x^{1/7})^6} + \frac{6}{7(x^{1/7})^{13}} = \frac{(x^{1/7})^7+6}{7(x^{1/7})^{13}}$$

$$f'(x)=0 \iff x+6=0 \iff x=-6$$

⇒ Critical numbers: $\boxed{-6}$ (no need to include 0, b/c in this case 0 is not in the domain of f!)

The absolute min/max values of a continuous f on $[a,b]$ occur either at a critical # or at an endpoint.

The Closed Interval Method:

To find the absolute min & max values of a continuous function f on a closed interval $[a,b]$ ← Don't test outside of $[a,b]$!

1. Find all critical numbers $c \in (a,b)$ of f.
2. Find $f(c)$ at all critical numbers $c \in (a,b)$. (Test @ critical pts.)
3. Find $f(a)$ and $f(b)$. (Test @ endpoints)
4. The largest/smallest value in Steps 2&3 is the absolute max/min.

Examples: Find the absolute min & max of f on given interval:

① $f(t) = t\sqrt{7-t}$ on $[1,6]$

$$f'(t) = \sqrt{7-t} + t \cdot \frac{-1}{2\sqrt{7-t}} = \sqrt{7-t} - \frac{t}{2\sqrt{7-t}} = \frac{2(7-t)-t}{2\sqrt{7-t}} = \frac{14-3t}{2\sqrt{7-t}}$$

$\Rightarrow$ Critical numbers of f: $t = \frac{14}{3}$ and $t = 7$ → not in $(1,6)$

belongs to $(1,6)$

$f\left(\frac{14}{3}\right) = \frac{14}{3}\sqrt{7-\frac{14}{3}} = \frac{14}{3}\sqrt{\frac{7}{3}}$ ← MAX

Test @ endpoints $\begin{cases} f(1) = \sqrt{6} & \leftarrow \text{MIN} \\ f(6) = 6 \end{cases}$

② $g(x) = \frac{1}{x-3}$ on $[-5,-4]$

$g'(x) = \frac{-1}{(x-3)^2}$ Only critical # of g is $x=3$ (where g' dne) but $3 \notin (-5,-4)$ so we only need to test @ endpoints.

$g(-5) = \frac{-1}{8}$ ← MAX

$g(-4) = \frac{-1}{7}$ ← MIN